Sampling and Reconstruction of Periodic Piecewise Polynomials Using Sinc Kernel

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SUMMARY We address a problem of sampling and reconstructing periodic piecewise polynomials based on the theory for signals with a finite rate of innovation (FRI signals) from samples acquired by a sinc kernel. This problem was discussed in a previous paper. There was, however, an error in a condition about the sinc kernel. Further, even though the signal is represented by parameters, these explicit values are not obtained. Hence, in this paper, we provide a correct condition for the sinc kernel and show the procedure. The point is that, though a periodic sinc kernel, we provide a correct condition for the sinc kernel by parameters, these explicit values are not obtained. Hence, we also show a sampling theorem for FRI signals with derivatives of a generic known function.

key words: Piecewise polynomials, stream of Diracs, finite rate of innovation (FRI) signals, annihilating filter

1. Introduction

Assume that we have two band-limited signals \( s_1(t) \) and \( s_2(t) \). Each of them can be perfectly sampled and reconstructed by the well-known sampling theorem [1]. Then, let us consider a new signal \( s(t) \), which is given by \( s_1(t) \) until a certain time instant, but is given by \( s_2(t) \) after that. The signal \( s(t) \) is now not band-limited, and therefore in general, can not be sampled and reconstructed by the sampling theorem anymore. Signals which have such discontinuity can be easily found around us. For example, music is a stream of sound pieces. Between each pieces, there exists discontinuity. In 2D cases, images naturally have edges.

A frequently used signal model is a polynomial B-spline [2]. It is, however, not appropriate for modeling such signals because the polynomial B-spline assumes smoothness up to the level of its degree minus one. A simple extension of B-spline is piecewise polynomial, which does not enforce continuity at any level. Now, relevant questions arise: can we sample such signals so that they are perfectly reconstructed? Further, can we develop a concrete algorithm which allows perfect reconstruction?

A good news is that the recent developments around sampling theory for signals with a finite rate of innovation opened a door to the new stage for such signals with discontinuity [3]. Assume that \( \varphi_r(t) \) are \( R \) known functions. We consider signals whose parametric representation is given by

\[
\varphi(t) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{R-1} a_{k,r} \varphi_r(t - t_k), \tag{1}
\]

where \( t_k \) and \( a_{k,r} \) are unknown time instants and signal coefficients, respectively. Introducing a counting function \( C_f(t_k,t_b) \) that returns the number of \( t_k \) in \( f(t) \) over the interval \([t_a,t_b]\) and the corresponding \( a_{k,r} \), we can define a rate of innovation \( \rho \) as

\[
\rho = \lim_{T \to \infty} \frac{1}{T} C_f(-\tau/2, \tau/2). \tag{2}
\]

Definition 1: [3] A signal with a finite rate of innovation (FRI signal) is a signal whose parametric representation is given in Eq. (1) and with a finite \( \rho \), as defined in Eq. (2).

The FRI signals include bandlimited signals as a special case by setting \( R = 1 \) and \( \varphi_0(t) = \text{sinc}(t/T) \). Such FRI signals can be reconstructed by the classical sampling theorem. In this case, a sample every \( T \) is required. If \( \rho \) is less than \( 1/T \), however, then this sampling seems to be wasteful. Sampling frequency should be as close to \( \rho \) as possible, following the notion of Occam’s razor [4].

Vetterli et al. first addressed this problem. They showed using sinc kernel that \( \tau \)-periodic sequence of \( K \) weighted Diracs per period (\( \rho = 2K/\tau \)) can be perfectly reconstructed from \( 2K + 1 \) samples per \( T \) [3]. They divided the reconstruction problem into two steps: first the time instants \( t_k \) are retrieved by using the so-called annihilating filter method, which is the standard auto regressive (AR) linear filter for spectral estimation. Then, the coefficients \( a_k \) are obtained by solving the standard Vandermonde system \( (a_k) = a_k \) because \( R = 1 \). Gaussian kernel was also used to sample finite-length FRI signals. Noisy scenario was discussed in [5]. The reference [6] is a good survey along this line.

In contrast to these kernels of infinite support, Dragotti et al. used sampling kernels of finite support [7]. They showed that non-periodic FRI signals can
be perfectly reconstructed from samples obtained by a kernel that satisfies the Strang-Fix condition, such as B-spline functions [2]. In this case too, the annihilating filter technique played a central role to solve the problem. This study was extended to multidimensional signals [8], sampling using E-spline functions [9], [10], and reconstruction of piecewise sinusoidal signals [11]. An interesting application is exact line-edge extraction from images [12], [13].

In Section 3, we provide a correct condition under which a periodic stream of differentiated Diracs is perfectly reconstructed. This is the point of the paper. Our target cause it is the standard one used to avoid the so-called ‘aliases caused by low sampling frequency. Our target is not one-to-one. Therefore, to recover the parameters in the latter does not directly lead us to those in the former. To solve this problem, we use the average of the target signal which is available because of the sinc kernel. Second, signals are reconstructed by using Fourier series, which requires an infinite number of summation. In computer simulations, however, we have to truncate the infinite series. Even though the signal is represented by parameters, these explicit values are not obtained.

Hence, in this paper, we first provide a correct condition for the bandwidth of the sinc kernel. Second, we show an exact reconstruction procedure. This is not trivial because the mapping from a periodic piecewise polynomial to a periodic stream of differentiated Diracs is not one-to-one. Therefore, to recover the parameters in the latter does not directly lead us to those in the former. To solve this problem, we use the average of the target signal which is available because of the sinc sampling. Then, the parameters in the periodic piecewise polynomials are successfully obtained and the signal is perfectly reconstructed. We also show another sampling theorem for FRI signals in Eq. (1)

\[ f(t) = \sum_{k'=-\infty}^{\infty} f_0(t - k'\tau), \]

without loss of generality. Acquisition process of samples \( \{d_n\}_{n=0}^{N-1} \) of \( f(t) \) is modeled by the equation

\[ d_n = \int_{-\infty}^{\infty} f(t)\psi(t-nT)dt, \]

where \( T = \tau/N \). The sampling kernel \( \psi(t) \) in this paper is

\[ \psi(t) = B\text{sinc}(Bt), \]

where \( B > 0 \) is the bandwidth divided by \( \pi \) and

\[ \text{sinc}(t) = \begin{cases} \sin(\pi t)/\pi t & (t \neq 0), \\ 1 & (t = 0). \end{cases} \]

Let \( P = [B\tau/2] \), the maximum integer not exceeding \( B\tau/2 \). From Poisson’s summation formula

\[ B \sum_{k'=-\infty}^{\infty} \text{sinc}(B(t + k'\tau)) = \frac{1}{\tau} \sum_{p=-P}^{P} e^{-i2\pi pt/\tau}, \]

it follows that

\[ d_n = \sum_{p=-P}^{P} \hat{d}_p e^{i2\pi n\pi/N}, \]

where \( \hat{d}_p \) is the Fourier coefficient of \( f(t) \):

\[ \hat{d}_p = \frac{1}{\tau} \int_0^\tau f(t)e^{-i2\pi pt/\tau}dt. \]

Eq. (6) implies that the Fourier coefficients \( \hat{d}_p \) are related to the sinc kernel samples \( d_n \) by the inverse discrete Fourier transform. Therefore, the forward transform provides the Fourier coefficients from the samples, as

\[ \hat{d}_p = \frac{1}{N} \sum_{n=0}^{N-1} d_n e^{-i2\pi n\pi/N}. \]

Let \( \hat{d}_p(f^{(r)}(t)) \) be the Fourier coefficients of the \( r \)th derivative \( f^{(r)}(t) \) of \( f(t) \):

\[ \hat{d}_p(f^{(r)}(t)) = \frac{1}{\tau} \int_0^\tau f^{(r)}(t)e^{-i2\pi pt/\tau}dt. \]

It is easy to show that...
\[ \hat{d}_p(f^{(r)}) = \left( \frac{2p \pi}{\tau} \right)^r \hat{d}_p(f). \]  

Hence, we can get \( \hat{d}_p(f^{(r)}) \) from \( d_n \) by multiplying \((i2p\pi/\tau)^r\) to \( \hat{d}_p(f) \), which we can obtain by Eq. (7).

### 3. Derivative of Diracs

Before proceeding to the main result of the paper, we consider a periodic stream of differentiated Diracs, \( g(t) \), which is defined by

\[ g(t) = \sum_{k=-\infty}^{\infty} g_0(t - k\tau), \]

with

\[ g_0(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R_k-1} a_{kr} \delta^{(r)}(t - t_k), \]

where \( 0 \leq t_0 < t_1 < \cdots < t_{K-1} < \tau \) and \( R_k > 1 \).

Further, \( \delta^{(r)}(t) \) denotes the \( r \)-th derivative of Dirac \( \delta(t) \) defined by

\[ \int_{-\infty}^{\infty} \delta^{(r)}(t)\phi(t)dt = (-1)^r \int_{-\infty}^{\infty} \delta(t)\phi^{(r)}(t)dt = (-1)^r \phi^{(r)}(0), \]

where \( \phi(t) \) is an arbitrary function that has derivatives of any order and tends to zero more rapidly than any power of \( t \), as \( |t| \) tends to infinity [14]. The signal \( g(t) \) plays the central role in this paper. The \((R+1)\) derivative of periodic piecewise polynomial is given by \( g(t) \).

The convolution of \( g(t) \) with a known function \( \varphi(t) \) defines a generic FRI signal discussed in Section 6. Hence, reconstruction of \( g(t) \) from its sinc samples is the central problem in this paper.

The degrees of freedom per period of \( \tau \) of Eq. (10) are \( K \) for time instants and \( \tilde{K} = \sum_{k=0}^{K-1} R_k \) for coefficients. Thus, the rate of innovation is

\[ \rho = \frac{K + \tilde{K}}{\tau}. \]

Theorem 3 in [3] requests that \( B \) of the sinc kernel in Eq. (5) is greater or equal to the rate of innovation: \( B \geq \rho \). This assumption seems, however, not sufficient for perfect reconstruction of \( g(t) \) as follows:

**Theorem 1:** Assume that \( B \) in Eq. (5) satisfies

\[ B \geq 2\tilde{K}/\tau, \]

and that

\[ N \geq 2P + 1 \]

with \( P = \lfloor B\tau/2 \rfloor \). Then, the sinc kernel samples \( \{d_n\}_{n=0}^{N-1} \) in Eq. (4) are a sufficient characterization of the \( \tau \)-periodic stream of differentiated Diracs, \( g(t) \).

To be self-contained, let us show a proof of the theorem.

**(Proof)** The Fourier coefficients \( \hat{d}_p \) of \( g(t) \) in Eq. (10) are given by

\[ \hat{d}_p = \frac{1}{\tau} \sum_{k=0}^{K-1} \sum_{r=0}^{R_k-1} a_{kr} \left( \frac{2p \pi}{\tau} \right)^r e^{-i2p\pi t_k/\tau}, \]

which, by letting \( \tilde{a}_{kr} = (i2\pi/\tau)^r a_{kr} \) and \( u_k = e^{-i2\pi t_k/\tau} \), can be simplified into

\[ \hat{d}_p = \sum_{k=0}^{K-1} \sum_{r=0}^{R_k-1} \tilde{a}_{kr} p^r u_k^p. \]

The sequence \( \hat{d}_p \) can be annihilated by the filter \( \{h_0, h_1, \ldots, h_{\tilde{K}}\} \) whose \( z \)-transform is

\[ H(z) = \prod_{k=0}^{\tilde{K}} (1 - u_k z^{-1}) \]

with \( R_k \) poles at \( z = u_k \), as shown in Appendix A in [3]. We can see immediately \( h_0 = 1 \). The annihilating equation

\[ \hat{d}_p + h_1 \hat{d}_{p-1} + \cdots + h_{\tilde{K}} \hat{d}_{p-\tilde{K}} = 0 \]

for \( p = 0, \ldots, \tilde{K} - 1 \) yields the matrix expression

\[ Uh = -d, \]

where

\[ U = \begin{pmatrix} \hat{d}_{-1} & \hat{d}_{-2} & \cdots & \hat{d}_{-\tilde{K}} \\ \hat{d}_0 & \hat{d}_{-1} & \cdots & \hat{d}_{-\tilde{K}+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{d}_{\tilde{K}-2} & \hat{d}_{\tilde{K}-3} & \cdots & \hat{d}_{-1} \end{pmatrix}, \]

and

\[ h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{\tilde{K}} \end{pmatrix}, \quad d = \begin{pmatrix} \hat{d}_0 \\ \hat{d}_1 \\ \vdots \\ \hat{d}_{\tilde{K}-1} \end{pmatrix}. \]

Eq. (16) requires \( 2\tilde{K} \) Fourier coefficients \( \hat{d}_p \), which are available via Eq. (7) from \( \{d_n\}_{n=0}^{N-1} \) because Eqs. (12) and (13) imply \( N \geq 2\tilde{K} + 1 \). Eq. (16) is a standard Yule-Walker system and has a unique solution since \( t_k \) are distinct. The factorization of the filter coefficients as in Eq. (15) yields \( u_k = e^{-i2\pi t_k/\tau} \), which lead to the locations \( t_k \).

Once \( u_k \) are obtained, Eq. (14) yields a linear matrix equation with respect to \( \tilde{a}_{kr}^* \):

\[ V\tilde{a} = \tilde{d}, \]

where \( V \) is a matrix.
By using \( K \) polynomials of degree less or equal to \( R \)

\[
s_k(t) = \sum_{r=0}^{R} a_{k,r} t^r \quad (k = 0 \sim K - 1), \quad (17)
\]

let us define \( \varphi_k(t) \) as

\[
\varphi_k(t) = \begin{cases} 
  s_k(t) & (t_k < t < t_{k+1}), \\
  0 & \text{(otherwise)},
\end{cases}
\]

for \( k = 0 \sim K - 2 \), or

\[
\varphi_{K-1}(t) = \begin{cases} 
  s_{K-1}(t+\tau) & (0 \leq t < t_0), \\
  s_{K-1}(t) & (t_{K-1} < t < \tau), \\
  0 & \text{(otherwise)}.
\end{cases}
\]

Then, a periodic piecewise polynomial \( f(t) \) of degree \( R \) is defined by Eq. (3) with

\[
f_0(t) = \sum_{k=0}^{K-1} \varphi_k(t). \quad (18)
\]

The degrees of freedom per period are \( K \) from the locations and \( \tilde{K} = (R+1)K \) from the coefficients. Hence, the rate of innovation is

\[
\rho = \frac{(R + 2)K}{\tau}.
\]

Even though \( \rho \) is a finite value, the piecewise polynomials do not seem to be the signal with a finite rate of innovation, because Eq. (18) is different from Eq. (1). The following lemma, however, relates the piecewise polynomials to the FRI signals:

**Lemma 1:** The \((R+1)\) derivative of \( f_0(t) \) is a stream of differentiated Diracs given by

\[
f_0^{(R+1)}(t) = g_0(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R} a_{k,r} \delta^{(r)}(t-t_k), \quad (19)
\]

with

\[
a_{k,r} = \begin{cases} 
  s_0^{(R-r)}(t_0) - s_{K-1}^{(R-r)}(t_0 + \tau) & (k = 0), \\
  s_k^{(R-r)}(t_k) - s_{k-1}^{(R-r)}(t_k) & (k = 1 \sim K - 1).
\end{cases}
\]

**Proof** As well as Eq. (11), the derivative of \( \varphi_k(t) \) in the distribution sense is given as

\[
\int_{-\infty}^{\infty} \varphi_k(t) \phi(t) dt = \int_{-\infty}^{\infty} \varphi_k(t) \phi(t) dt = 0 - \int_{t_k}^{t_{k+1}} s_k(t) \phi'(t) dt
\]

\[
= -[s_k(t) \phi(t)]_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} s'_k(t) \phi(t) dt
\]

\[
= s_k(t_k) \phi(t_k) - s_k(t_{k+1}) \phi(t_{k+1}) + \int_{-\infty}^{\infty} s'_k(t) \{u(t-t_k) - u(t-t_{k+1})\} \phi(t) dt,
\]
which implies
\[
\varphi_k(t) = s_k(t_k)\delta(t - t_k) - s_k(t_{k+1})\delta(t - t_{k+1}) + s_k'(t(t - t_k) - s_k'(t(t - t_{k+1})).
\]
Note that \(s_k'(t)\) is the differentiation of \(s_k(t)\) in the standard sense. By repeating this computation \(R+1\) times, we have \(s_k^{(R+1)}(t) = 0\) and therefore
\[
\varphi_k^{(R+1)}(t) = \sum_{r=0}^{R} \left( s_k^{(R-r)}(t_k)\delta^{(r)}(t - t_k) - s_k^{(R-r)}(t_{k+1})\delta^{(r)}(t - t_{k+1}) \right).
\]

Summing up Eq. (20) in terms of \(k\) yields Eq. (19). □

The signal in Eq. (19) clearly falls within the framework of the signals with a finite rate of innovation. Further, Eq. (8) means that we can get \(\hat{d}_p(f^{(R+1)})\) by multiplying \((2\pi i/\tau)^{R+1}\) to \(\hat{d}_p(f)\). Therefore, applying the technique in Section 4 to \(\hat{d}_p(f^{(R+1)})\) allows us to obtain \(\alpha_{k,r}\) in Eq. (19).

After that, we have to retrieve the \(\alpha_{k,r}\) from \(a_{k,r}\).

Generally, a polynomial of degree \(R\) is uniquely determined if every value of its \(r\) derivative from \(r = 0\) to \(R\) is provided. Eq. (19) seems to satisfy this condition. The sum of \(a_{k,R}\), however, always vanishes, because of the periodicity:
\[
\sum_{k=0}^{K-1} a_{k,R} = 0
\]
Hence, there are infinitely many periodic piecewise polynomials that are mapped to Eq. (19). We need more information to uniquely determine Eq. (18) from Eq. (19). To this end, we use the average of the signal, which is available from the Fourier coefficient for \(p = 0\), i.e., \(d_0\). These observations lead us to the following sampling theorem for the periodic piecewise polynomials:

**Theorem 2:** Assume that \(B\) in Eq. (5) satisfies Eq. (12) and that Eq. (13) holds with \(P = \lfloor B_\tau/2 \rfloor\). Then, the samples \(\{d_n\}_{n=0}^{N-1}\) in Eq. (4) using sinc kernel are a sufficient characterization of the \(\tau\)-periodic piecewise polynomial \(f(t)\).

**(Proof)** Multiplying \((i2\pi \tau/\tau)^{(R+1)}\) to \(\hat{d}_p(f^{(R+1)})\) yields the Fourier coefficient \(\hat{d}_p(f^{(R+1)})\) of \(f^{(R+1)}(t)\), which is the periodic stream of differentiated Diracs in Eq. (19). This signal is reconstructed by applying the procedure in the proof of Theorem 1. To determine the coefficients \(\alpha_{k,r}\) in Eq. (17), we solve the following linear equation:
\[
a = W\alpha,
\]
where \(a\) and \(\alpha\) are vectors defined by
\[
a = \begin{pmatrix} a_0,0 \\ a_1,0 \\ \vdots \\ a_{K-1},0 \\ d_0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{K-1,R} \end{pmatrix},
\]
respectively, and \(W\) is a matrix depending on \(K\), \(R\), \(\tau\), and \(t_k\). An example of \(W\) for \(K = 4\) and \(R = 2\) is shown in Table 1. The size of \(W\) is \(\lfloor (R + 1)/K(R + 1) \rfloor \times \lfloor (R + 1)/K(R + 1) \rfloor\), not square. It is convenient to decompose \(W\) into \(R+1\) block matrices \(W_{r,r'}\), which for \(0 \leq r, r' \leq R\) are \(K \times K\) square matrices. The rest \(W_{r+1,r'}\) for \(0 \leq r' \leq R\) are \(K\) dimensional row vectors. The blocks are also shown in Table 1. Those for \(0 \leq r, r' \leq R-1\) are the zero matrix, while the blocks for \(r+1 \leq r' \leq R\) are cyclic matrices whose ranks are \(K-1\). Other matrices are of full-rank for \(t_k\)'s are distinct. Further, since it holds that
\[
\tau W_{R+1,0} = -(t_0, t_1, \ldots, t_{K-1})W_{R,0} + \tau(0, 0, 0, 0, 1),
\]
the \((R + 1)\)st row is independent from the other rows. Hence, \(W^T W\) is the full-rank \(K(R + 1) \times K(R + 1)\) square matrix, and has the inverse. Therefore, by multiplying the Moore-Penrose Generalized inverse \(W^{-1} = (W^T W)^{-1}W^T\) to Eq. (21) from the left, we can obtain \(a = (W^T W)^{-1}W^T a\), which yields our target signal. □

**Theorem 2** is a complete version of Theorem 4 in
[3]. That is, we showed the correct condition for $B$ so that the periodic piecewise polynomials are perfectly reconstructed. Further, we provided a concrete reconstruction procedure, which is illustrated in Figure 2.

In the simulation shown in [3], $K = 4$, $R = 1$, $\tilde{K} = 8$, $P = 50$ and $\tau = 1024$ were used. These parameters imply that $B$ is a value between $100/1024$ and $102/1024$, which is much greater than $2\tilde{K}/\tau = 16/1024$. Hence, the system (16) was successfully established. If a critical value was used, like $(4 + 16)/1024$, then the system could not be obtained.

Figure 3 shows a simulation result for sampling and reconstruction of a periodic piecewise polynomial of $\tau = 10$, $K = 4$, and $R = 2$. Then, $K = 12$ and $B$ must be greater or equal to 2.4, which we used as $B$. Since $P = 12$, we used the critical number of samples $N = 25$. The thick dashed (black) and thin solid (red) lines in (a) respectively show the target signal and reconstructed signal from samples shown by the bullet. The sampling functions for $n = 11$, 12, and 13 used in this simulation are shown in Figure (b). The reconstructed result was within the machine precision. The number $N = 25$ of samples used in this simulation is more than the number of unknown parameters, $K + \tilde{K} = 16$. To reduce the number of samples, we have to determine the locations $t_k$ more efficiently than using annihilating filter as in Eq. (15).

5. Derivatives of Generic Function

The result obtained in Section 3 can also be applied to the problem of sampling and reconstruction of FRI signals $f(t)$ in Eq. (3) with

$$f_0(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R-1} a_{k,r} \varphi^{(r)}(t - t_k),$$

where $\varphi(t)$ is a given function that has the Fourier transform $\hat{\varphi}(\omega)$. This signal $f(t)$ is the convolution of $g(t)$ with $\varphi(t)$:

$$f(t) = (\varphi * g)(t) = \int_{-\infty}^{\infty} \varphi(t')g(t - t')dt'.$$

Then, the Fourier coefficient $\hat{d}_p(f)$ of $f(t)$ is expressed using that of $g(t)$ and the Fourier transform $\hat{\varphi}(\omega)$ as

$$\hat{d}_p(f) = \hat{\varphi}(\frac{2\pi p}{\tau})\hat{d}_p(g),$$

where $\hat{d}_p(f)$ can be obtained from $d_n$ by Eq. (7). Hence,
as long as $\varphi(2\pi/n) \neq 0$, applying the technique in Section 3 to $d_p(g) = d_p(f)/\varphi(2\pi/n)$ enables us to retrieve unknown time instants and then the signal coefficients. Therefore, we have the following

**Theorem 3:** Assume that $B$ in Eq. (5) satisfies Eq. (12) and that Eq. (13) is true with $P = [B\tau/2]$. The function $\varphi(t)$ is assumed to satisfy $\varphi(2\pi n/\tau) \neq 0$ for $p = -P \sim P$. Then, the samples $\{d_n\}_{n=1}^N$ in Eq. (4) using the sinc kernel are a sufficient characterization of the $\tau$-periodic FRI signal $f(t)$ in Eq. (22).

For example, let $\varphi(t)$ be the centered cubic B-spline function as

$$\varphi(t) = \beta_3(t) = (\beta_0 + \beta_1 + \beta_0)(t),$$

where

$$\beta_0(t) = \begin{cases} 1 & (|t| < 0.5), \\ 0 & (|t| \geq 0.5). \end{cases}$$

It is easy to compute the derivative of B-spline of degree $p$ because of the relation

$$\frac{d\beta_p(t)}{dt} = \beta_{p-1}(t + 1/2) - \beta_{p-1}(t - 1/2)$$

shown in [2]. The parameters used in the simulation were $K = 4$, $R_k = 2$ for $k = 0 \sim 3$, and $\tau = 15$. Then, $K = 8$ and Theorem 3 requests that $B$ must be greater or equal to $16/15$, which we used as $B$. Since $P$ becomes $8$, $N$ have to be greater or equal to $17$, which we used as $N$. Because $\varphi(\omega) = \{\text{sinc}(\omega/2\pi)\}^4$, we can see that $\varphi(2\pi n/\tau) \neq 0$.

A simulation result is shown in Figure 4. The thick dashed (black) and thin solid (red) lines show the target and reconstructed signals, respectively. Note that the right end part around [12, 15] is completely flat and we know that this signal is not bandlimited. The bullets show the samples obtained using the sinc kernel. We can see that the reconstructed signal is again within the machine precision.

**6. Conclusion**

In this paper, we addressed the problem of sampling and reconstruction of periodic piecewise polynomials from samples obtained using the sinc kernel. Even though this problem was discussed in the previous paper, there was an error in a condition for the sinc kernel and only reconstruction procedure using Fourier series was shown. Hence, we first provided a correct condition for the sinc kernel, saying that we have to use a bandwidth more than twice of the number of unknown coefficients over period. Based on this result, we derived the sampling theorem for the periodic piecewise polynomials. We pointed out that, since the mapping from a periodic stream of differentiated Diracs to the periodic piecewise polynomial is not one-to-one, information other than the periodic stream of differentiated Diracs is necessary to uniquely recover the target signal. To this end, we used the average of the signal, which is available as the Fourier coefficient for $p = 0$. Then, the parameters in the piecewise polynomials are successfully obtained, and the signal was perfectly reconstructed. We further showed a sampling theorem for FRI signals with derivatives of a given function. Our future tasks include reduction of the number of samples by retrieving the time instants more efficiently.

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**References**


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